CS 58500 – Theoretical Computer Science Toolkit

Lecture 19 (04/21)

Boolean Function Analysis (IV)

https://ruizhezhang.com/course_spring_2026.html

Today's Lecture

- Influence and Isoperimetric Inequalities for the Hypercube
- KKL Theorem
- Unifying Talagrand and KKL

Recall the Tensorization of Variance

Theorem (Efron-Stein). Suppose $X_1, \ldots, X_n$ are independent random variables. Let $Z = f(X_1, \ldots, X_n)$. Then

$$\mathrm{Var}[Z] \le \mathbb{E}\left[\sum_{i=1}^{n} \mathrm{Var}_i[Z]\right]$$

where $\mathrm{Var}_i[Z](x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \coloneqq \mathrm{Var}_{X_i}[f(x_1, \ldots, x_{i-1}, X_i, x_{i+1}, \ldots, x_n)]$

In Boolean function analysis, we define the influence of f as:

$$I_i(f) \coloneqq \mathbb{E}_{x_{-i}}[\mathrm{Var}_i[f](x)], \qquad I_f \coloneqq \sum_{i\in[n]} I_i(f)$$

- If we define $\partial_i f \coloneqq f - \mathbb{E}_i[f]$, then $I_i(f) = \mathbb{E}_{x_{-i}}\left[\mathbb{E}_i[f - \mathbb{E}_i[f]]^2\right] = \|\partial_i f\|_2^2$

Influence

For $f : \{0,1\}^n \to \{0,1\}$,

$$\partial_i f(x) = f(x) - \frac{1}{2}\big(f(x) + f(x + e_i)\big) = \frac{1}{2}\big(f(x) - f(x + e_i)\big)$$

$$I_i(f) = \frac{1}{4}\Pr[f(x) \neq f(x + e_i)]$$

Example:

- Parity: $I_i = \frac{1}{4}$ and $I_{\text{Parity}} = \frac{n}{4}$

$$I_f = \frac{1}{4}\mathbb{E}\big[s_f(x)\big]$$

Influence

For a general $f : \{0,1\}^n \to \mathbb{R}$,

$$I_i(f) = \frac{1}{4}\mathbb{E}\left[\left(f(x) - f(x+e_i)\right)^2\right]$$
$$= \frac{1}{4}\mathbb{E}\left[\left(2\sum_{S \ni i} \hat{f}(S)\chi_S(x)\right)^2\right]$$
$$= \sum_{S \ni i} \hat{f}(S)^2$$

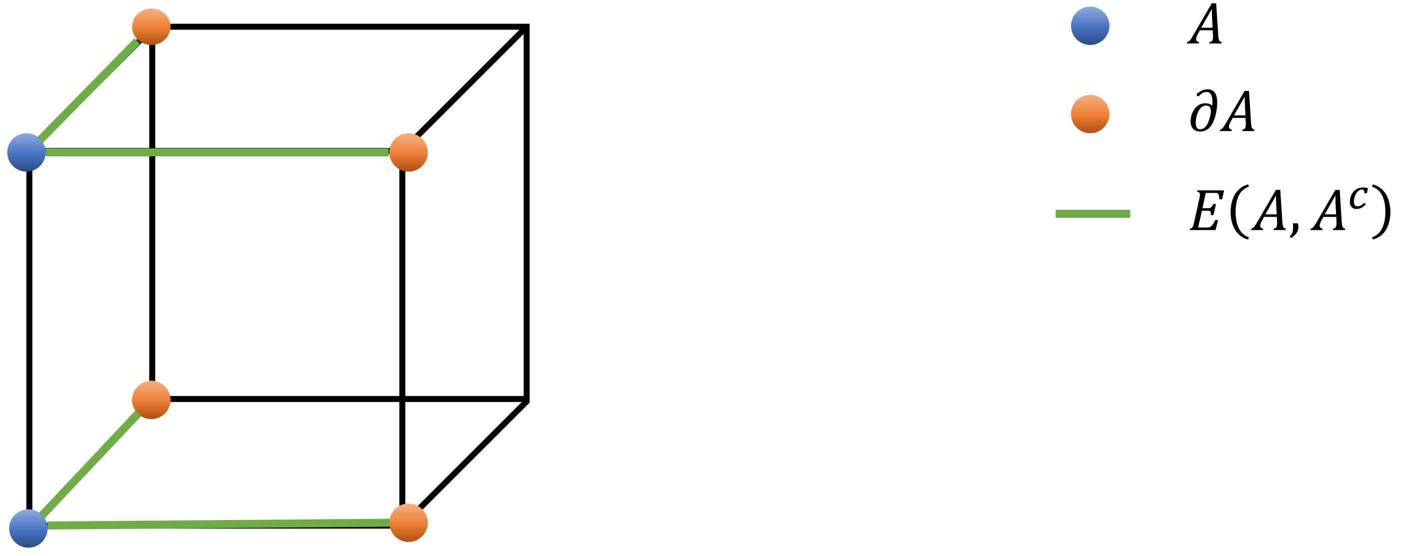
And

$$I_f = \sum_{i=1}^{n} \sum_{S \ni i} \hat{f}(S)^2 = \sum_{S} |S| \hat{f}(S)^2$$

Isoperimetric Inequalities for the Hypercube

Let $G = (V, E)$ be an undirected graph. For $A \subseteq V$, the edge boundary of A is the set $E(A, A^c)$ of edges with one endpoint is A and one in $V \backslash A$. The vertex boundary of A is the set

$$\partial A = \{v \in A^c : v \text{ has a neighbor in } A\}$$



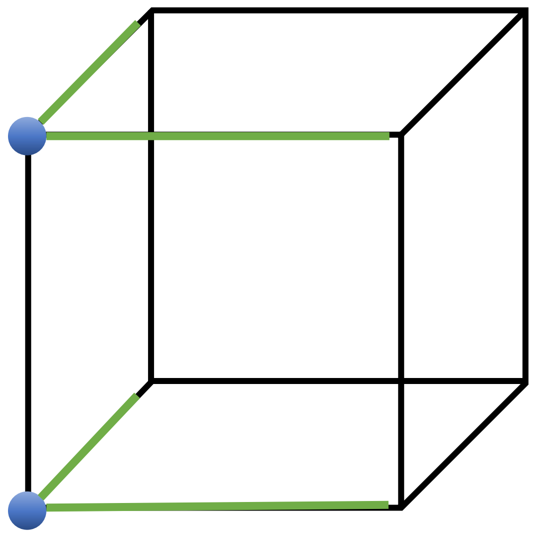
Isoperimetric Inequalities for the Hypercube

Theorem (Harper's edge isoperimetric inequality). Every subset A of the vertices of the hypercube $\mathcal{Q}_n$ satisfies

$$|E(A, A^c)| \geq |A| \log \frac{2^n}{|A|}$$

with equality when A is a subcube.

- $n = 3$
  - $|A| = 4, |E(A, A^c)| \geq 4 * \log \frac{8}{4} = 4$
  - $|A| = 2, |E(A, A^c)| \geq 2 * \log \frac{8}{2} = 4$

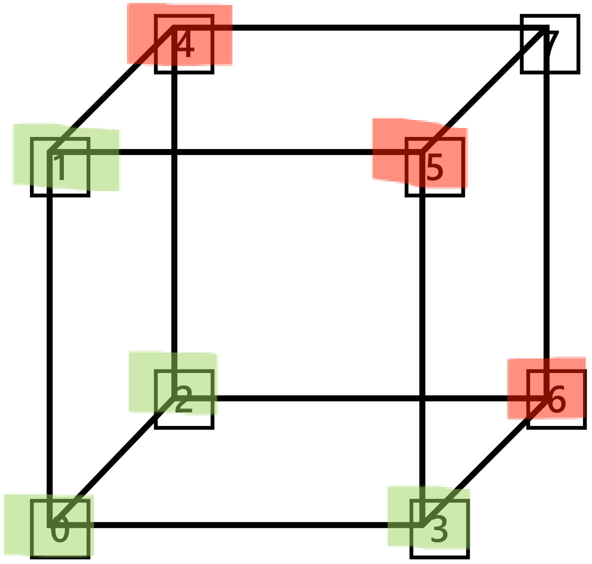


Isoperimetric Inequalities for the Hypercube

Theorem (Harper's vertex isoperimetric inequality). Every subset A of the vertices of the hypercube Q_n with $\binom{n}{\le r} \le |A| < \binom{n}{\le r+1}$ satisfies

$$|A \cup \partial A| \ge \binom{n}{\le r+1}$$

where $\binom{n}{\le r} := \sum_{i=0}^{r} \binom{n}{i} = |B_r(x)|$ (the size of a Hamming ball of radius r)



Define the simplicial order of the hypercube:

$$000 < 100 < 010 < 001 < 110 < 101 < 011 < 111$$

Theorem (Harper's vertex isoperimetric inequality). For any A,

$$|\partial A| \ge |\partial L_{|A|}|,$$

where $L_{|A|}$ is the initial segment in simplicial orderings of length $|A|$

Influence and Isoperimetric Inequalities for the Hypercube

For a Boolean function $f:\{0,1\}^n \to \{0,1\}$, we can define:

- Vertex boundary: $V_f \coloneqq \left\{x \in \{0,1\}^n : s_f(x) > 0\right\}$
- Edge boundary: $E_f \coloneqq \{(x, x+e_i) : f(x) \neq f(x+e_i)\}$

Claim. $I_f = \left|E_f\right|/2^{n+1}$

Proof.

- Notice that $2\left|E_f\right| = \sum_x s_f(x)$
- Thus,

$$I_f = \frac{1}{4}\mathbb{E}\left[s_f(x)\right] = 2^{-(n+2)} \cdot 2\left|E_f\right| = 2^{-(n+1)}\left|E_f\right|$$

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Influence and Isoperimetric Inequalities for the Hypercube

Theorem (Harper's edge isoperimetric inequality**).** Every $f:\{0,1\}^n \to \{0,1\}$ satisfies

$$I_f \geq \frac{1}{2}\mathbb{E}[f]\log\frac{1}{\mathbb{E}[f]}$$

Proof.

- $f:\{0,1\}^n \to \{0,1\} \quad \Longleftrightarrow \quad f = \mathbf{1}_A$ where $A \coloneqq \{x : f(x) = 1\}$
- $|E_f| = |E(A, A^c)| \geq |A| \log\frac{2^n}{|A|} = 2^n \mathbb{E}[f]\log(\mathbb{E}[f]^{-1})$
- $I_f = |E_f| \cdot 2^{-(n+1)}$

$\blacksquare$

Influence and Isoperimetric Inequalities for the Hypercube

Poincare inequality:

$$\frac{|E_f|}{2^{n+1}} = I_f \geq \mathrm{Var}[f]$$

Proof.

- $I_f = \sum_S |S| \hat{f}(S)^2 \geq \sum_{S \neq \emptyset} \hat{f}(S)^2 = \mathrm{Var}[f]$

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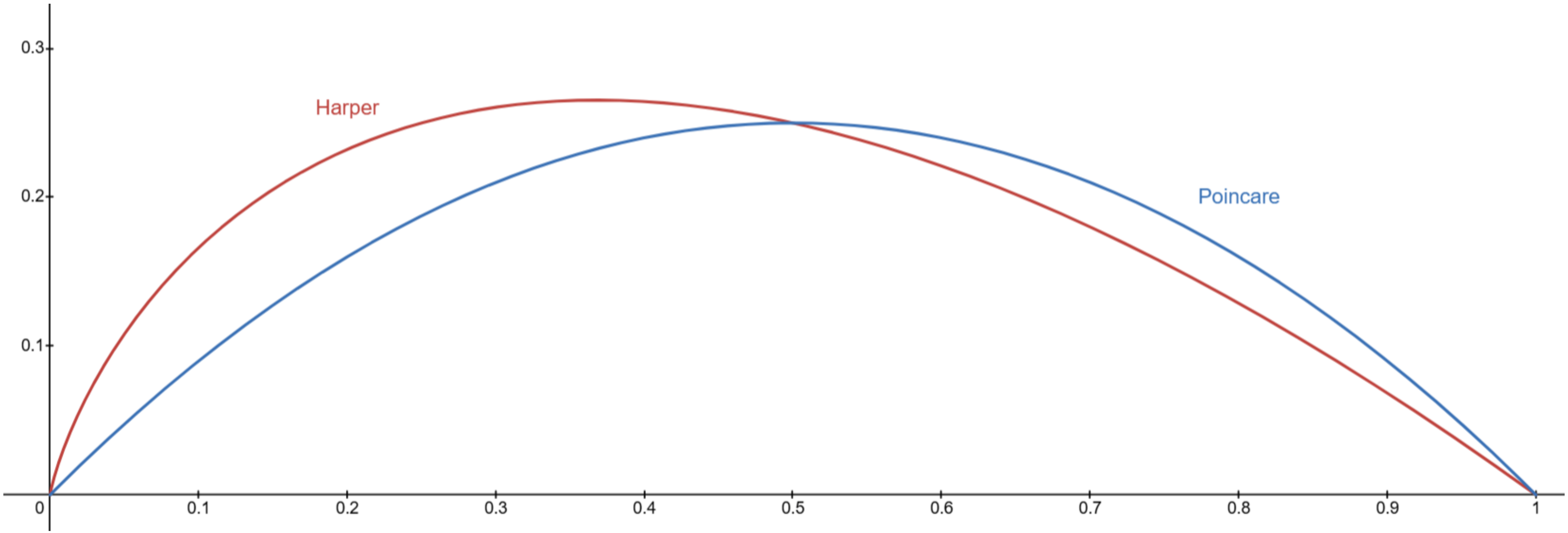
Influence and Isoperimetric Inequalities for the Hypercube

$$I_f \geq \max\left\{\mathrm{Var}[f], \frac{1}{2}\,\mathbb{E}[f]\log\frac{1}{\mathbb{E}[f]}\right\}$$



Influence and Isoperimetric Inequalities for the Hypercube

What about the vertex boundary $V_f := \left\{x \in \{0,1\}^n : s_f(x) > 0\right\}$?

Examples:

- Dictator $f(x) = x_1$: $V_f = \{0,1\}^n$, $\left|V_f\right| = 2^n$
- Majority $f(x) = \mathrm{Majority}(x)$:

$$V_f = \left\{x \in \{0,1\}^n : |x|_1 = \frac{n}{2}\right\}, \qquad \left|V_f\right| = \binom{n}{n/2} = \mathcal{O}\left(\frac{2^n}{\sqrt{n}}\right)$$

This is optimal by the Kruskal-Katona theorem

Theorem (Margulis '74). For every $f: \{0,1\}^n \to \{0,1\}$,

$$\frac{\left|V_f\right|}{2^n} \cdot \frac{\left|E_f\right|}{2^n} \simeq \frac{\left|V_f\right|}{2^n} \cdot I_f \geq \Omega(\mathrm{Var}[f]^2)$$

- When the vertex boundary is small, edge boundary (or influence) must be very large

Influence and Isoperimetric Inequalities for the Hypercube

Michel Talagrand considered a new quantity $\mathbb{E}\left[\sqrt{s_f(x)}\right]$ (Talagrand boundary of f) and proved several results

- The basic one is $\mathbb{E}\left[s_f(x)^{1/2}\right] \geq \Omega(\mathrm{Var}[f])$
- By Cauchy-Schwarz inequality,

$$\mathbb{E}\left[s_f(x)^{1/2}\right] = \mathbb{E}\left[s_f(x)^{1/2} \cdot \mathbf{1}_{s_f(x)>0}\right] \leq \mathbb{E}\left[s_f(x)\right]^{1/2} \cdot \Pr\left[s_f(x) > 0\right]^2 \simeq \left(\frac{|V_f|}{2^n} \cdot I_f\right)^{1/2}$$

recovering Margulis's theorem

Theorem (Talagrand '93). For all non-constant $f\colon \{0,1\}^n \to \{0,1\}$,

$$\mathbb{E}\left[s_f(x)^{1/2}\right] \geq \Omega\left(\mathrm{Var}[f] \log \frac{1}{\mathrm{Var}[f]}\right)$$

- Tight for Majority function

Today's Lecture

- Influence and Isoperimetric Inequalities for the Hypercube
- KKL Theorem
- Unifying Talagrand and KKL

KKL Theorem

Theorem (Kahn-Kalai-Linial '88). For $f: \{0,1\}^n \to \{0,1\}$, let $I_{\max} := \max_i I_i(f)$. Then,

$$\mathrm{Var}[f] \lesssim \frac{I_f}{\log(1/I_{\max})}$$

Since $I_f \leq nI_{\max}$, we have

$$I_{\max} \geq \Omega\left(\frac{\log n}{n}\mathrm{Var}[f]\right)$$

- KKL is another strengthening of Poincare:

$$I_f \gtrsim \min_i \log\left(\frac{1}{I_i(f)}\right) \cdot \mathrm{Var}[f]$$

- If all individual influences of f being small, then it gains a large factor

KKL Theorem

Theorem (Kahn-Kalai-Linial '88). For $f:\{0,1\}^n \to \{0,1\}$, let $I_{\max} \coloneqq \max_i I_i(f)$. Then,

$$\mathrm{Var}[f] \lesssim \frac{I_f}{\log(1/I_{\max})}$$

Proof.

- Recall that $\mathrm{Var}[f] = \sum_{S\neq\emptyset} \hat{f}(S)^2$ and $I_f = \sum_S |S|\hat{f}(S)^2$
- Bounding high-degree coefficients:

$$I_f \geq k \cdot \sum_{|S|>k} \hat{f}(S)^2, \qquad \Rightarrow \quad \sum_{|S|>k} \hat{f}(S)^2 \leq \frac{I_f}{k}$$

- Bounding low-degree coefficients:

$$\sum_{|S|\leq k} \left|\hat{f}(S)\right|^2 \leq \sum_{i\in[n]} \sum_{i\in S, |S|\leq k} \left|\hat{f}(S)\right|^2 = \sum_{i\in[n]} \left\|(\partial_i f)^{\leq k}\right\|_2^2$$

where $\partial_i f(x) = \sum_{i\in S} \hat{f}(S)\chi_S(x)$ and $(\partial_i f)^{\leq k}$ takes the degree $\leq k$ terms

Detour: Hypercontractivity

Theorem. For $f: \{0,1\}^n \to \mathbb{R}$ and $k > 0$,

1) For $2 \le q < \infty$,

$$\left\|f^{\le k}\right\|_q \le (q-1)^{\frac{k}{2}} \|f\|_2$$

2) For $1 \le p \le 2$,

$$\left\|f^{\le k}\right\|_2 \le e^{\left(\frac{2}{p}-1\right)k} \|f\|_p$$

- 2) can be simplified: let q be such that $\frac{1}{q} + \frac{1}{p} = 1$. Then by Hölder's and 1),

$$\left\|f^{\le k}\right\|_2^2 = \langle f^{\le k}, f \rangle \le \left\|f^{\le k}\right\|_q \|f\|_p \le (q-1)^{\frac{k}{2}} \left\|f^{\le k}\right\|_2 \|f\|_p = (p-1)^{-\frac{k}{2}} \left\|f^{\le k}\right\|_2 \|f\|_p$$

$$\left\|f^{\le k}\right\|_2 \le (p-1)^{-\frac{k}{2}} \|f\|_p$$

KKL Theorem

Theorem (Kahn-Kalai-Linial '88). For $f: \{0,1\}^n \to \{0,1\}$, let $I_{\max} := \max_i I_i(f)$. Then,

$$\mathrm{Var}[f] \lesssim \frac{I_f}{\log(1/I_{\max})}$$

Proof.

- Bounding low-degree coefficients:

$$\sum_{|S|\le k} |\hat{f}(S)|^2 \le \sum_{i\in[n]} \sum_{i\in S, |S|\le k} |\hat{f}(S)|^2 = \sum_{i\in[n]} \|(\partial_i f)^{\le k}\|_2^2$$

- By hypercontractivity,

$$\sum_{|S|\le k} |\hat{f}(S)|^2 \le \sum_{i\in[n]} \|(\partial_i f)^{\le k}\|_2^2 \le \sum_{i\in[n]} 3^k \|\partial_i f\|_{4/3}^2$$

- **Claim:** $\mathbb{E}[|\partial_i f|^p] \le \mathbb{E}[|\partial_i f|] = 2\mathbb{E}[|\partial_i f|^2] = 2I_i(f)$

- $\sum_{i\in[n]} \|\partial_i f\|_{4/3}^2 \simeq \sum_{i\in[n]} I_i(f)^{3/2} \le I_{\max}^{1/2} \cdot \sum_{i\in[n]} I_i(f) = I_{\max}^{1/2} \cdot I_f$

KKL Theorem

Theorem (Kahn-Kalai-Linial '88). For $f:\{0,1\}^n \to \{0,1\}$, let $I_{\max} := \max_i I_i(f)$. Then,

$$\mathrm{Var}[f] \lesssim \frac{I_f}{\log(1/I_{\max})}$$

Proof.

- Thus, we have

$$\mathrm{Var}[f] = \sum_{|S|>k} \hat{f}(S)^2 + \sum_{|S|\le k} \left|\hat{f}(S)\right|^2 \lesssim \frac{I_f}{k} + 3^k I_{\max}^{1/2} \cdot I_f$$

- Taking $k \simeq \log \frac{1}{I_{\max}}$ proves the theorem

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KKL Theorem

Theorem (Kahn-Kalai-Linial '88**).** For $f:\{0,1\}^n \to \{0,1\}$, let $I_{\max} \coloneqq \max_i I_i(f)$. Then,

$$I_{\max} \geq \Omega\left(\frac{\log n}{n}\mathrm{Var}[f]\right)$$

KKL is tight due to the tribes function:

$$f(x) = \bigvee_{i\in[m]} \bigwedge_{j\in[k]} x_{i,j}\,, \qquad k = \log n - \log\log n\,, m = n/k$$

- $\mathrm{Var}[f] \geq \Omega(1)$
- $I_i(f) \leq \big(1 - o(1)\big)\frac{\ln n}{2n}$

Today's Lecture

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Unifying Talagrand and KKL

$$\mathbb{E}\left[s_f(x)^{1/2}\right] \geq \Omega\left(\mathrm{Var}[f]\log\frac{1}{\mathrm{Var}[f]}\right)$$

$$I_{\max} \geq \Omega\left(\frac{\log n}{n}\mathrm{Var}[f]\right)$$

Talagrand showed that there exists an $\alpha \in (0,1/2]$ such that

$$\mathbb{E}\left[s_f(x)^{1/2}\right] \gtrsim \mathrm{Var}[f]\cdot\log^{\frac{1}{2}-\alpha}\left(\frac{1}{\mathrm{Var}[f]}\right)\cdot\log^{\alpha}\left(1+\frac{1}{\sum_i I_i(f)^2}\right)$$

- $\mathbb{E}\left[s_f(x)^{1/2}\right] \leq \mathbb{E}\left[s_f(x)\right]^{1/2} = I_f^{1/2}$
- $\frac{1}{\sum_i I_i(f)^2} \geq \frac{1}{I_{\max} I_f}$
- It recovers KKL if $\alpha = 1/2$

$\Rightarrow$

$$I_{\max} \geq e^{-\mathcal{O}\left(\left(\frac{I_f}{\mathrm{Var}[f]}\right)^{\frac{1}{2\alpha}}\right)}$$

Unifying Talagrand and KKL

$$\mathbb{E}\left[s_f(x)^{1/2}\right] \geq \Omega\left(\mathrm{Var}[f] \log \frac{1}{\mathrm{Var}[f]}\right)$$

$$I_{\max} \geq \Omega\left(\frac{\log n}{n} \mathrm{Var}[f]\right)$$

Talagrand showed that there exists an $\alpha \in (0,1/2]$ such that

$$\mathbb{E}\left[s_f(x)^{1/2}\right] \gtrsim \mathrm{Var}[f] \cdot \log^{\frac{1}{2}-\alpha}\left(\frac{1}{\mathrm{Var}[f]}\right) \cdot \log^{\alpha}\left(1 + \frac{1}{\sum_i I_i(f)^2}\right)$$

Theorem (Eldan-Gross '20**).**

$$\mathbb{E}\left[s_f(x)^{1/2}\right] \gtrsim \mathrm{Var}[f] \cdot \sqrt{\log\left(1 + \frac{1}{\sum_i I_i(f)^2}\right)}$$

Unified Proof by Eldan-Kindler-Lifshitz-Minzer

If we define $\nabla f: \{0,1\}^n \to \{-1,0,1\}^n$ to be $\nabla f(x) = \big(\partial_1 f(x), \partial_2 f(x), \dots, \partial_n f(x)\big)$, then

$$\|\nabla f(x)\|_2 = \sqrt{s_f(x)}$$

Since $|\partial_i f| = |f(x_{i=0}) - f(x_{i=1})|/2 = \left|\sum_{i \in S} \hat{f}(S)\chi_{S-\{i\}}\right|$, we have

$$\mathbb{E}[|\partial_i f|] = \mathbb{E}\left[\left|\sum_{i \in S} \hat{f}(S)\chi_{S-\{i\}}\right|\right] \geq \left|\sum_{i \in S} \hat{f}(S)\mathbb{E}\left[\chi_{S-\{i\}}\right]\right| = \left|\hat{f}(\{i\})\right|$$

Thus,

$$\mathbb{E}[\|\nabla f(x)\|_2] \geq \left\|(\mathbb{E}[|\partial_i f|])_{i \in [n]}\right\|_2 \geq \left\|\left(\left|\hat{f}(\{i\})\right|\right)_{i \in [n]}\right\|_2 = \left(\sum_i \hat{f}(\{i\})^2\right)^{1/2} = \sqrt{\mathbf{W}^1[f]}$$

That is,

$$\mathbb{E}\left[s_f(x)^{1/2}\right] \geq \sqrt{\mathbf{W}^1[f]} \geq \mathbf{W}^1[f]$$

Unified Proof

Suppose that we knew f has significant weight on level-d ($d > 1$); can we use the same logic to lower bound $\mathbb{E}[\|\nabla f\|_2]$?

Random Restriction

If f has sizable mass around level d, then $f_{\bar{J} \to z}$ has sizable weight around level pd

Corollary. Let $d \in \mathbb{N}$ and $p \in [0,1]$ be such that $pd \geq 10$. Suppose that $f: \{0,1\}^n \to \{-1,1\}$, and let (J, z) be a p-random restriction. Define the weight around level d to be

$$\mathbf{W}^{\approx d}[f] := \sum_{d \leq k < 2d} \mathbf{W}^k[f]$$

Then

$$\mathbb{E}_{J,z}\left[\mathbf{W}^{\approx pd}[f_{\bar{J} \to z}]\right] \geq \Omega(\mathbf{W}^{\approx d}[f])$$

- If we take $p = \frac{1}{2d}$, then $\mathbb{E}_{J,z}\left[\mathbf{W}^1[f_{\bar{J} \to z}]\right] \gtrsim \mathbf{W}^{\approx d}[f]$

Unified Proof

Lemma. Let $d \in \mathbb{N}$, and $f:\{0,1\}^n \to \{0,1\}$. Then $\mathbb{E}[\|\nabla f\|_2] \gtrsim \sqrt{d}\mathbf{W}^{\approx d}[f]$

Proof.

- Let (J, z) be a $1/2d$-random restriction. Then, for any $x \in \{0,1\}^n$, we have

$$\mathbb{E}_J\left[\left\|\nabla f_{\bar{J}\to z}(x_J)\right\|_2\right] \le \sqrt{\mathbb{E}_J\left[\left\|\nabla f_{\bar{J}\to z}(x_J)\right\|_2^2\right]} = \sqrt{\mathbb{E}_J\left[\sum_j \left|\partial_j f(x)\right|^2 \cdot 1_{j\in J}\right]} = \frac{\|\nabla f\|_2}{\sqrt{2d}}$$

Thus,

$$\mathbb{E}[\|\nabla f\|_2] \gtrsim \sqrt{d}\,\mathbb{E}_{J,z}\left[\mathbb{E}_x\left[\left\|\nabla f_{\bar{J}\to z}(x_J)\right\|_2\right]\right] \ge \sqrt{d}\,\mathbb{E}_{J,z}\left[\mathbf{W}^1[f_{\bar{J}\to z}]\right] \gtrsim \sqrt{d}\mathbf{W}^{\approx d}[f]$$

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- If we apply the lemma for $d, 2d, 4d, \cdots$ and sum them up, we can further bootstrap:

$$\mathbb{E}[\|\nabla f\|_2] \gtrsim \sqrt{d}\mathbf{W}^{\ge d}[f]$$

Unified Proof

Then, we can apply the known Fourier tail bounds to prove Talagrand and Eldan-Gross:

- If f has small variance, then most of its Fourier mass lies above level $d = \Omega(\log(1/\mathrm{Var}[f]))$
- If f has small influences, then most Fourier mass lies above level $d = \Omega(\log(1/\sum_i I_i(f)^2))$

Unified Proof

$$\mathbb{E}\left[s_f(x)^{1/2}\right] \geq \Omega\left(\mathrm{Var}[f]\log\frac{1}{\mathrm{Var}[f]}\right)$$

Proof.

Case 1: $\mathrm{Var}[f] \geq 2^{-16}$:

$$\mathbb{E}[\|\nabla f\|_2] \gtrsim \mathbf{W}^{\geq 1}[f] = \mathrm{Var}[f]$$

Case 2: $\mathrm{Var}[f] < 2^{-16}$:

$$\sum_{0<|S|\leq d} \hat{f}(S)^2 \leq \left\|f^{\leq d}\right\|_2^2 = \left\langle f^{\leq d}, f\right\rangle \leq \left\|f^{\leq d}\right\|_4 \|f\|_{4/3}$$

- Hypercontractivity: $\left\|f^{\leq d}\right\|_4 \leq 3^{d/2}\left\|f^{\leq d}\right\|_2 \leq 3^{d/2}\|f\|_2$

$$\left\|f^{\leq d}\right\|_4 \|f\|_{4/3} \leq 3^{d/2}\|f\|_2\|f\|_{4/3} = 3^{d/2}\mathbb{E}[f]^{5/4}$$

- We may assume $\mathbb{E}[f] \leq 1/2$, and thus, $\mathbb{E}[f] \leq 2\mathrm{Var}[f]$
- If we take $d = \frac{1}{8}\log(1/\mathrm{Var}[f])$, then $\sum_{0<|S|\leq d} \hat{f}(S)^2 \leq 0.9\mathrm{Var}[f]$ and $\mathbf{W}^{\geq d}[f] \geq 0.1\mathrm{Var}[f]$ ∎

Unified Proof

$$\mathbb{E}[s_f(x)^{1/2}] \gtrsim \mathrm{Var}[f] \cdot \sqrt{\log\left(1 + \frac{1}{\sum_i I_i(f)^2}\right)}$$

Proof.

Theorem (Keller-Kindle '13**).** There are $c_1, c_2 > 0$ such that for any $f: \{0,1\}^n \to \{0,1\}$,

$$\mathbf{W}^{\leq c_1 \log(1/M[f])}[f] \leq M[f]^{c_2}$$

where $M[f] \coloneqq \sum_i I_i(f)^2$

Case 1: $M[f] \geq \mathrm{Var}[f]^{2/c_2}$

$$\mathrm{Var}[f] \cdot \log^{1/2}\left(1 + \frac{1}{M[f]}\right) \lesssim \mathrm{Var}[f] \log \frac{1}{\mathrm{Var}[f]}$$

Case 2: $M[f] < \mathrm{Var}[f]^{2/c_2}$

- Take $d = c_1 \log(1/M[f])$, and we have

$$\mathbf{W}^{\leq d}[f] \leq M[f]^{c_2} \leq \mathrm{Var}[f]^2 \leq 0.5\mathrm{Var}[f]$$

∎

Poincare

$$I_f \gtrsim \mathrm{Var}[f]$$

Harper

$$I_f \gtrsim \mathbb{E}[f] \log \frac{1}{\mathbb{E}[f]}$$

Margulis

$$I_f \gtrsim \mathrm{Var}[f] \frac{2^n}{|V_f|}$$

Talagrand

$$\mathbb{E}\left[s_f(x)^{1/2}\right] \geq \Omega\left(\mathrm{Var}[f] \log \frac{1}{\mathrm{Var}[f]}\right)$$

KKL

$$I_f \gtrsim \mathrm{Var}[f] \log \frac{1}{I_{\max}}$$

Eldan-Gross

$$\mathbb{E}\left[s_f(x)^{1/2}\right] \gtrsim \mathrm{Var}[f] \cdot \sqrt{\log\left(1 + \frac{1}{\sum_i I_i(f)^2}\right)}$$

Final Story: Aaronson-Ambainis Conjecture

Conjecture. Any function $f: \{-1,1\}^n \to [-1,1]$ of degree d satisfies

$$I_{\max} \geq \mathrm{poly}\left(\frac{\mathrm{Var}[f]}{d}\right)$$

- If any quantum algorithm has quantum query complexity q, then there exists a classical algorithm that makes $\mathrm{poly}(q)$ queries and succeeds on average over the inputs
- KKL $\Longrightarrow$ for Boolean f, $I_{\max} \gtrsim \frac{\log n}{n} \mathrm{Var}[f]$.
  Since $D(f) \leq \deg(f)^4$ and f can only depend on at most $2^{D(f)}$ many variables, it gives an exponential bound for AA conjecture
- (O'Donnell-Saks-Schrammand-Servedio '05) proves that $I_{\max} \gtrsim \frac{\mathrm{Var}[f]}{D(f)} \geq \frac{\mathrm{Var}[f]}{d^4}$, which implies a polynomial bound for AA conjecture for Boolean functions
- Read Chap. 10 of Thomas Rothvoss's lecture notes for more discussion